Integrable particle systems and Macdonald processes

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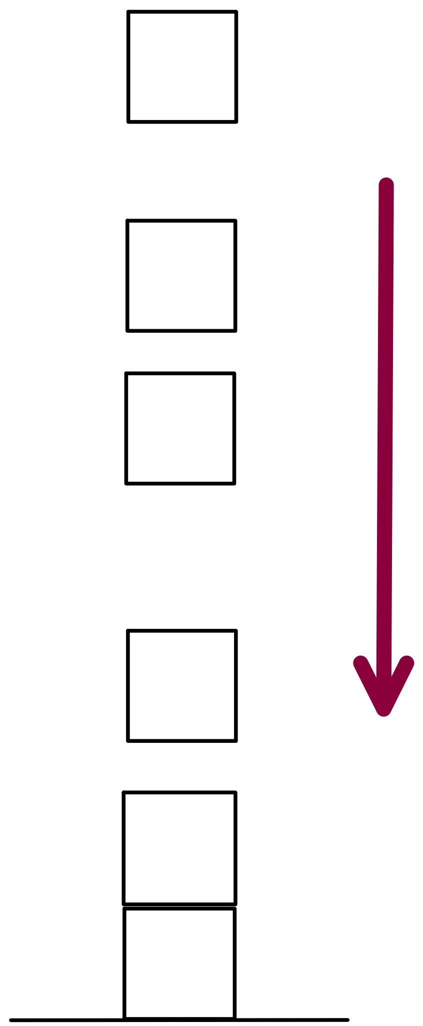
<u>Lecture 1</u>

- Introduction to integrable probability
- The Kardar-Parisi-Zhang universality class
- The totally asymmetric simple exclusion process (TASEP)
- The GUE corner process
- Warren's dynamics and continuous space TASEP

What is integrable probability?



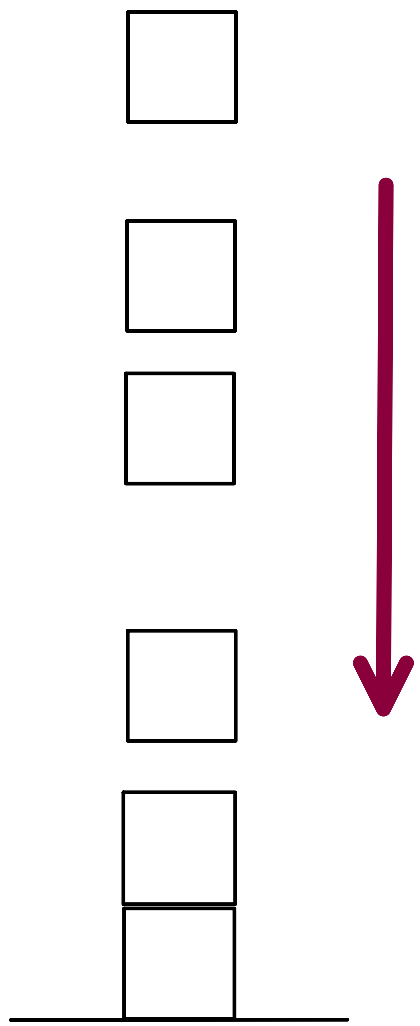
Imagine you are building a tower out of standard square blocks that fall down at random time moments.
How tall will it be after a large time T?

It is natural to expect that

Height = const $\cdot$ T + random fluctuations

What can one say about the fluctuations?

What is integrable probability?



A simple integrable example: The time is discrete, each second a new block falls with probability 1/2 (independently of what happened before). Then

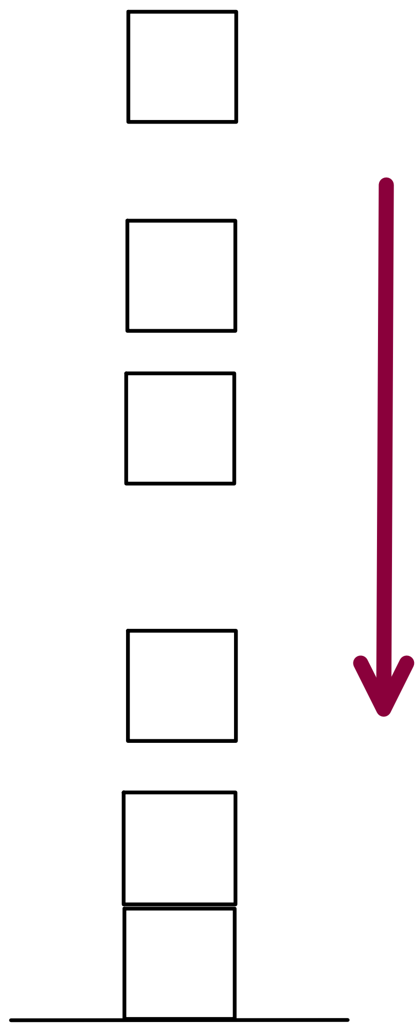
$$\mathbb{P}\{\text{height} = n \ \text{ after time } T\} = \frac{1}{2^T}\,\frac{T!}{n!\,(T-n)!}$$

Theorem *[De Moivre 1738], [Laplace 1812]*

$$\lim_{T\to\infty} \mathbb{P}\left\{\text{height}(T) \le \frac{T}{2} + \frac{s}{2}\cdot T^{1/2}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-x^2/2}\, dx$$

Ex: Prove Stirling's approximation use to prove above result.

What is integrable probability?



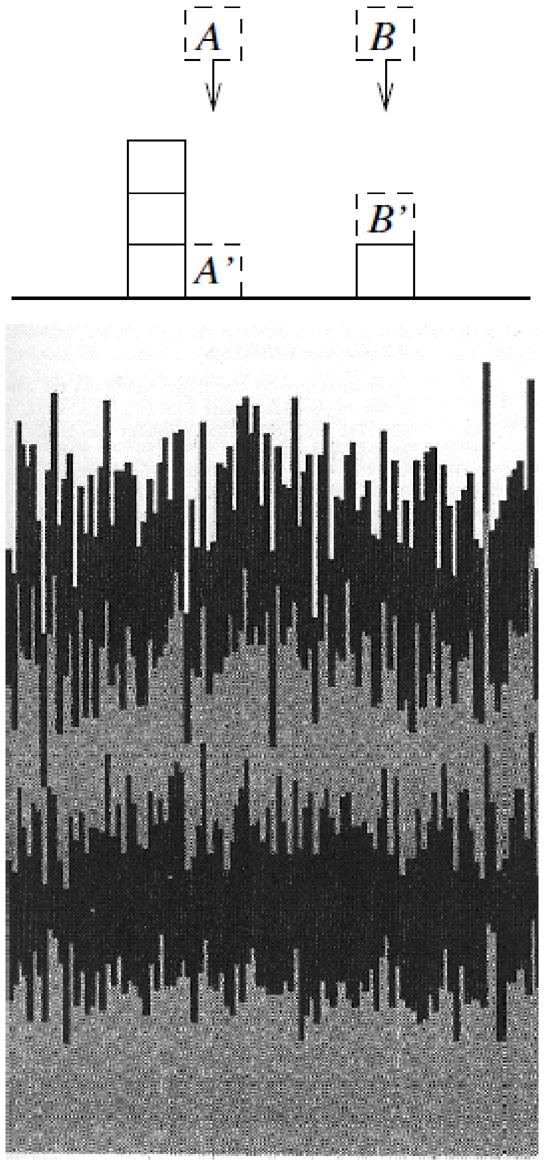
Universality principle: For a broad class of different possibilities for randomness, the size and distribution of the fluctuations must be the same (up to scaling constants)

This is the Central Limit Theorem, first proved by Lyapunov in 1901.

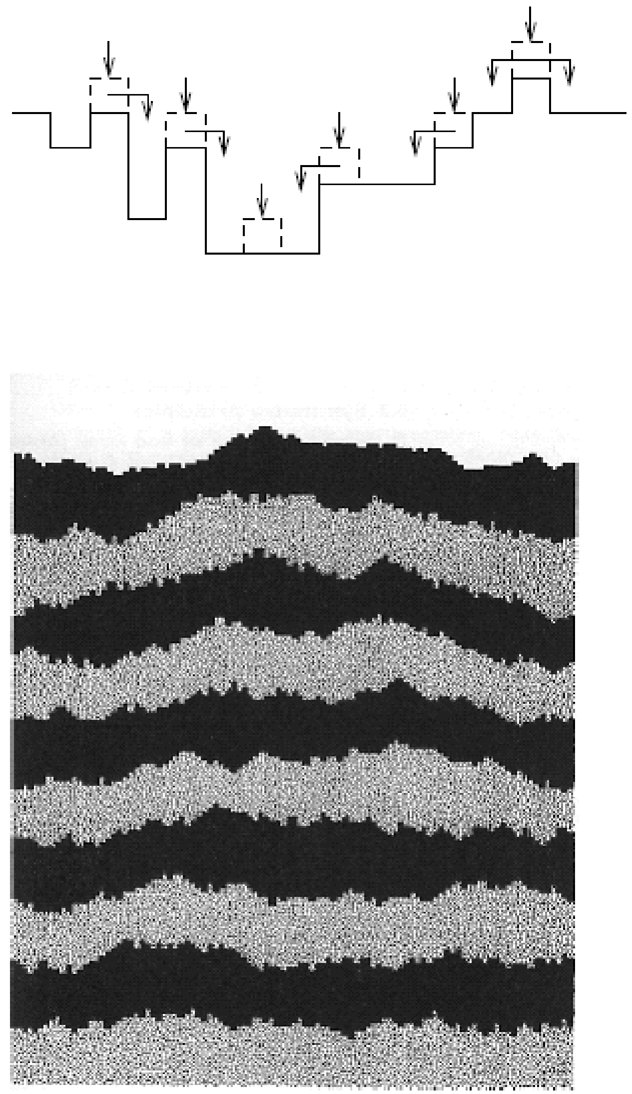
An integrable example predicts the behavior of the whole universality class.

Three models of random interface growth in (1+1)d

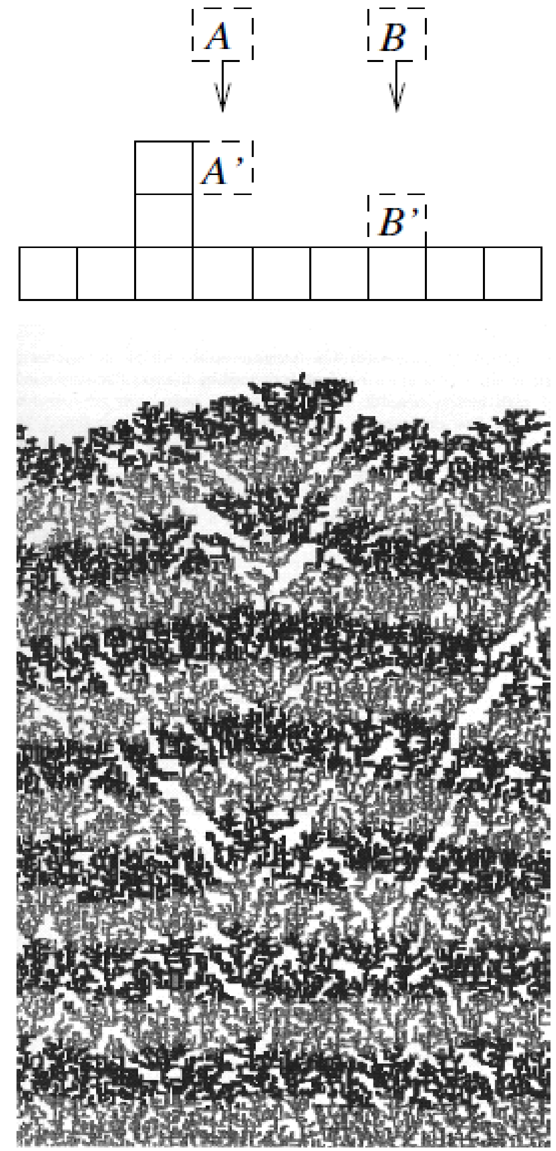
Random deposition



Random deposition with relaxation



Ballistic deposition

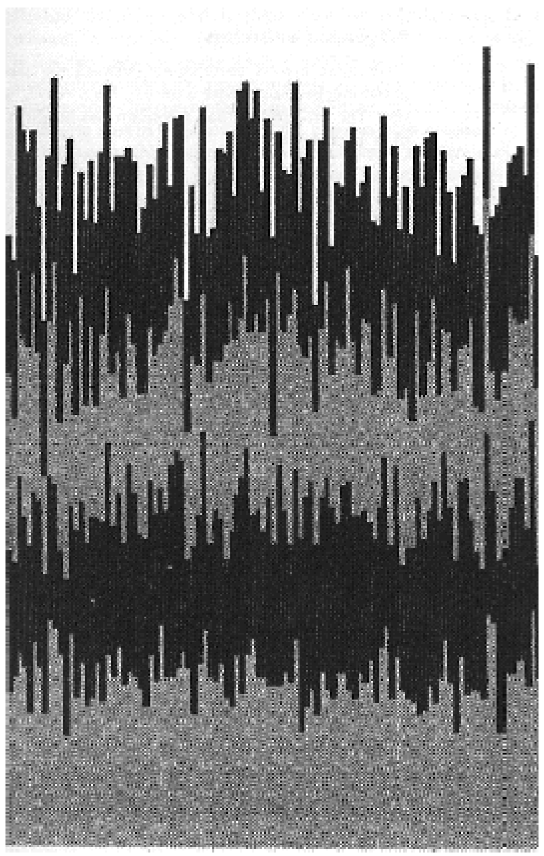


Three models of random interface growth in (1+1)d

Classical CLT

$\partial_t h = \eta(x,t)$

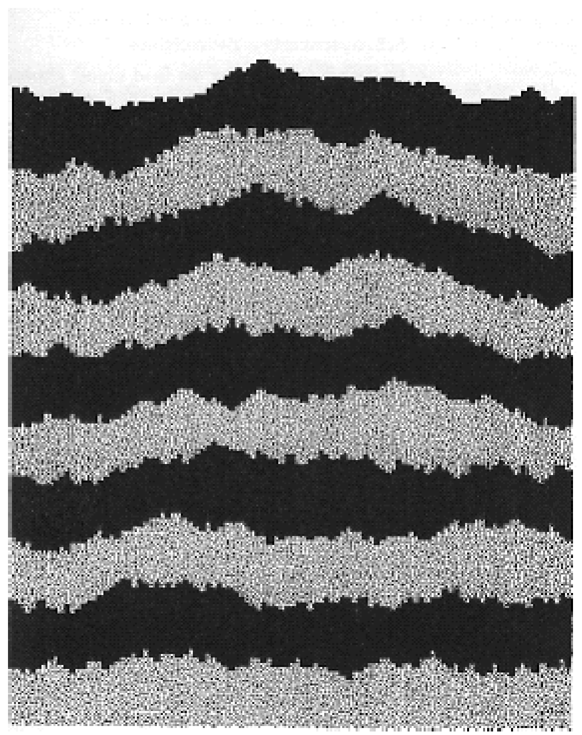
$t^{\frac{1}{2}}$ fluctuations



Edwards-Wilkinson eq.

$\partial_t h = \nu\, \partial_x^2 h + \eta(x,t)$

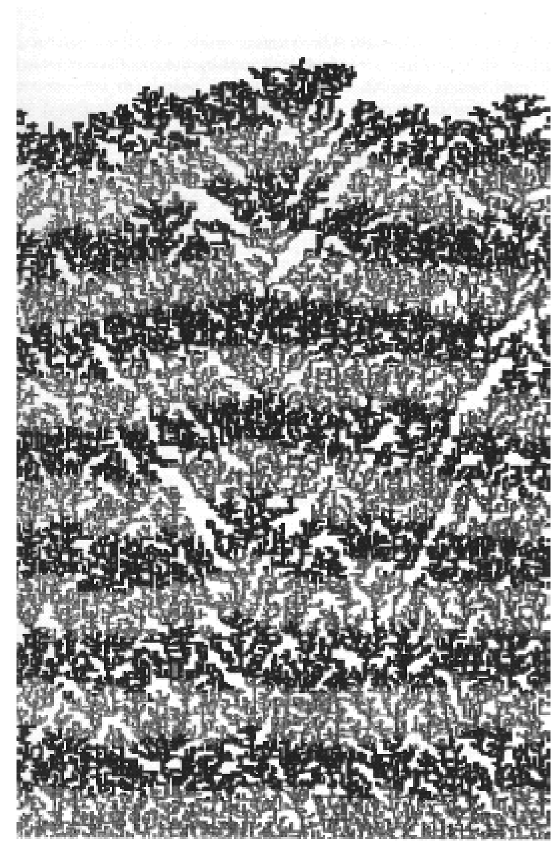
$t^{\frac{1}{4}}$ fluctuations



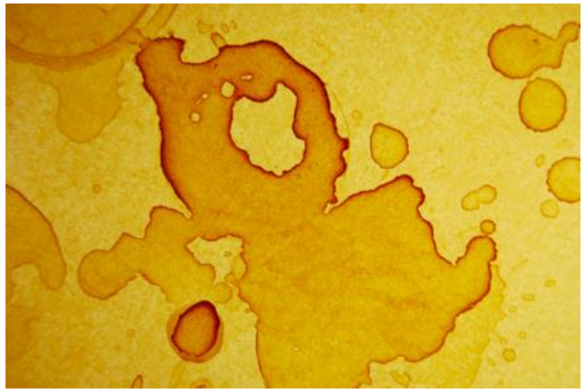
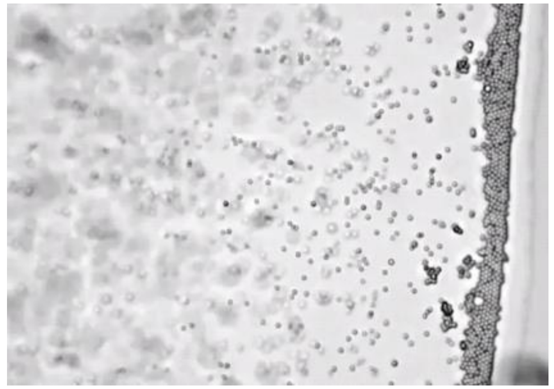
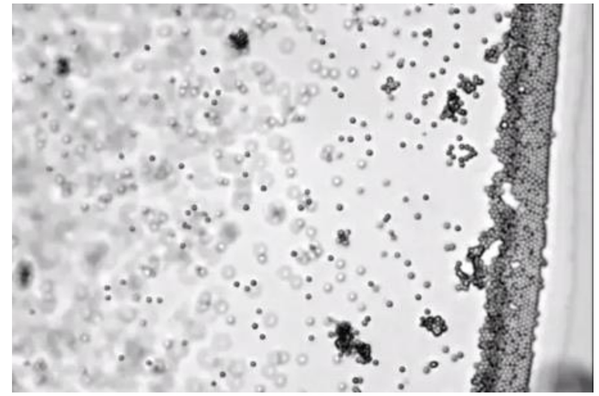
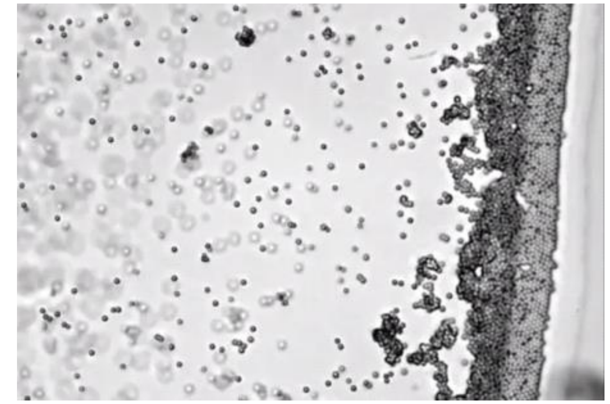
Kardar-Parisi-Zhang eq.

$\partial_t h = \nu\, \partial_x^2 h + \lambda(\partial_x h)^2 + \eta$

$t^{\frac{1}{3}}$ fluctuations



Experimental example: coffee ring effect

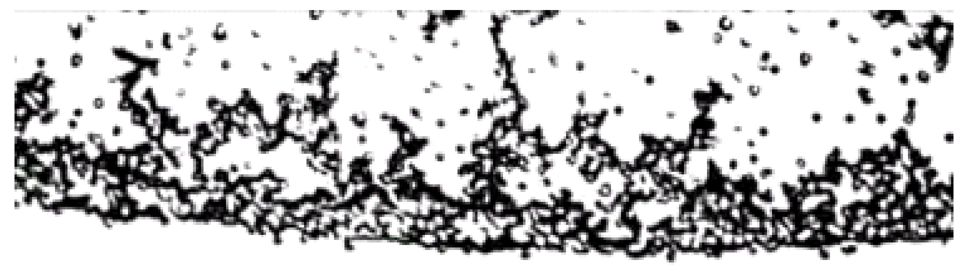





Perfectly round particles:

$t^{1/2}$ fluctuations, CLT statistics

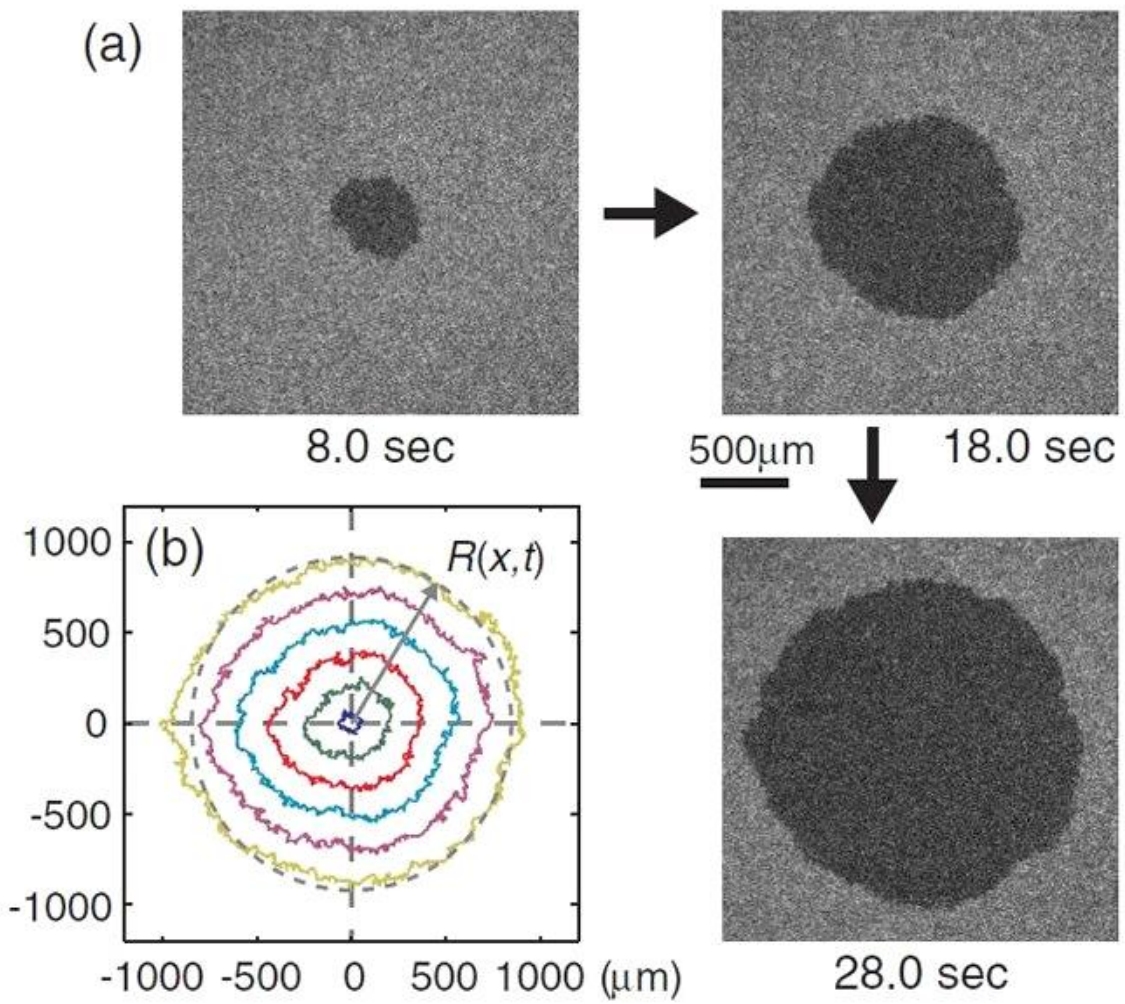
Slightly elongated particles:

$t^{1/3}$ fluctuations, KPZ statistics

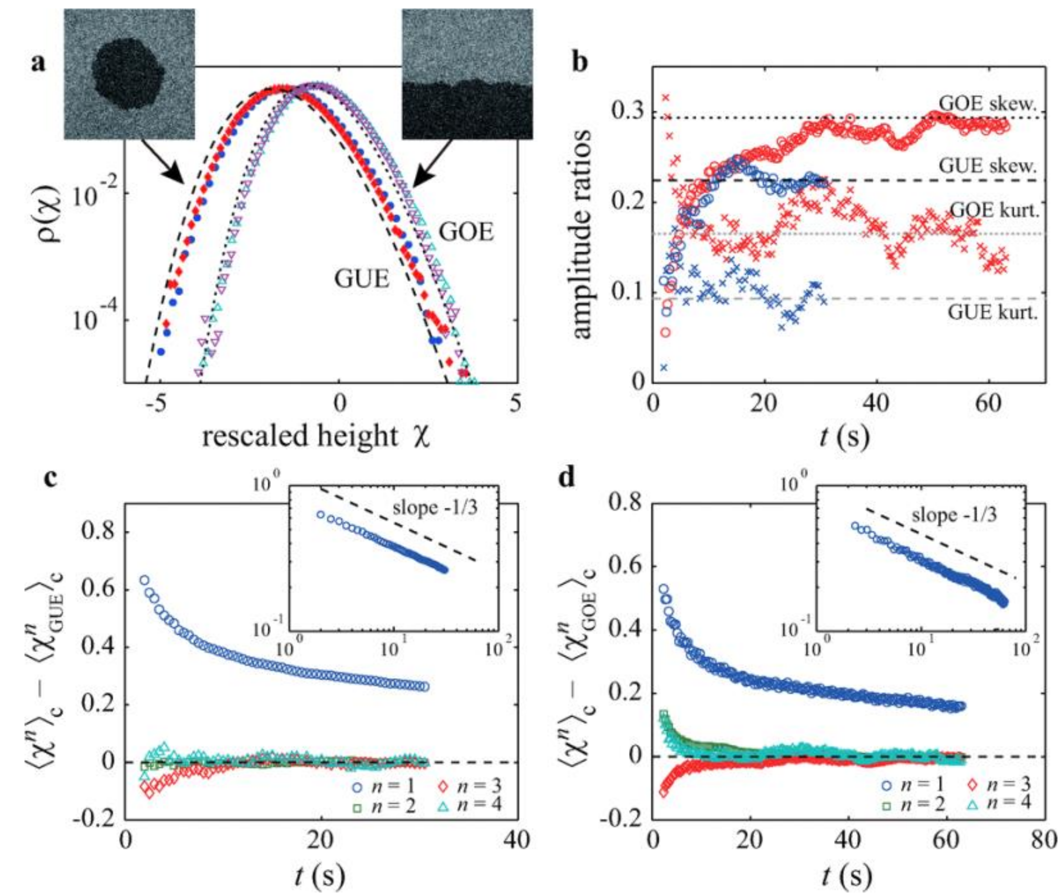


[Yunker-Lohr-Still-Borodin-Durian-Yodh, PRL 2013]

Experimental example: Disordered liquid crystals growth

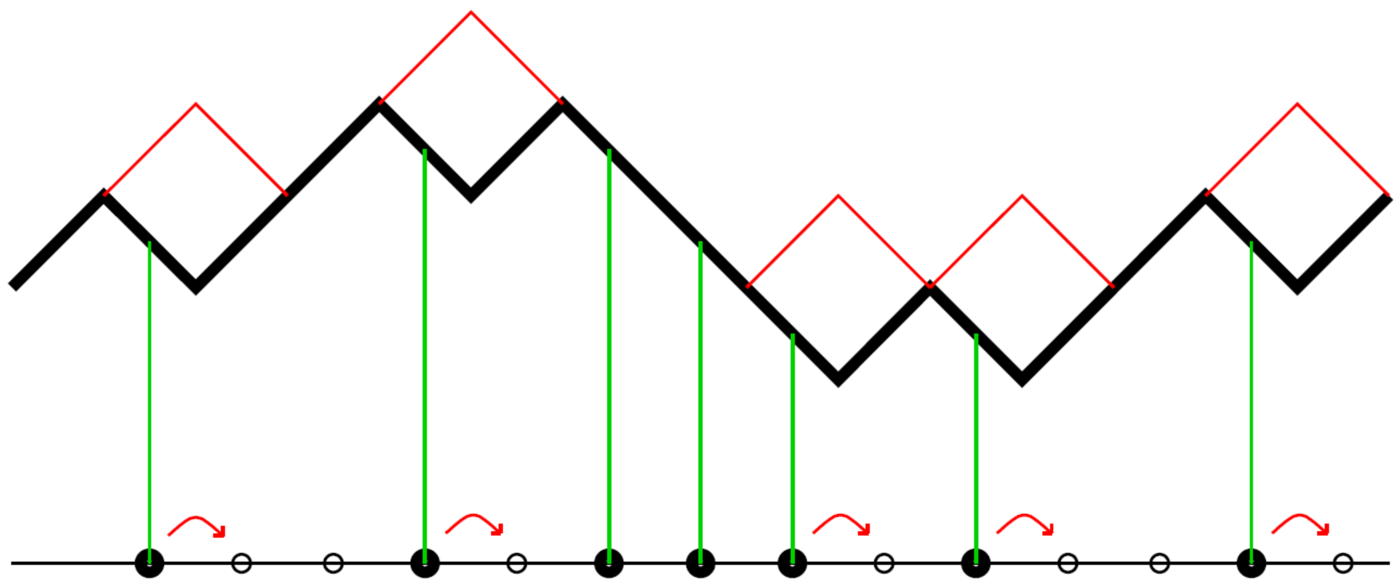


$t^{1/3}$ fluctuations, geometry dependent KPZ statistics



[Takeuchi, Sano PRL 2010],
[Takeuchi, Sano, Sasamoto, Spohn PRL 2011]

Totally Asymmetric Simple Exclusion Process (TASEP): An integrable random interface growth model



TASEP animation

Red boxes are added independently at rate 1. Equivalently, particles with no right neighbor jump independently with waiting time distributed as $e^{-x}dx$.

Ex: Construct the particle process for TASEP on $\mathbb{Z}$.

Kardar-Parisi-Zhang (KPZ) universality class

TASEP is an integrable representative of the (conjectural) KPZ universality class of growth models in 1+1 dimensions that is characterized by

- Locality of growth (no long-range interaction)
- A smoothing mechanism (a.k.a. relaxation)
- Lateral growth (speed of growth depends non-linearly on slope)

If the speed of growth does not depend on the slope than the model is in the Edwards-Wilkinson class with different fluctuations ($t^{1/4}$ instead of $t^{1/3}$ and Gaussian distributions)

TASEP - hydrodynamic limit

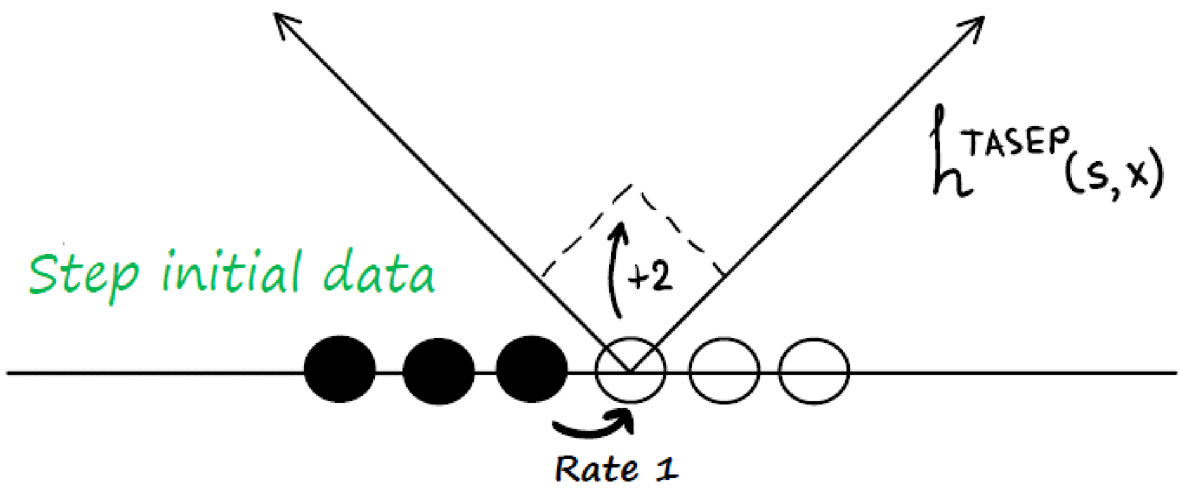
At large time, over distances comparable with time (macroscopic or hydrodynamic limit), the evolution of the interface is deterministic with high probability.

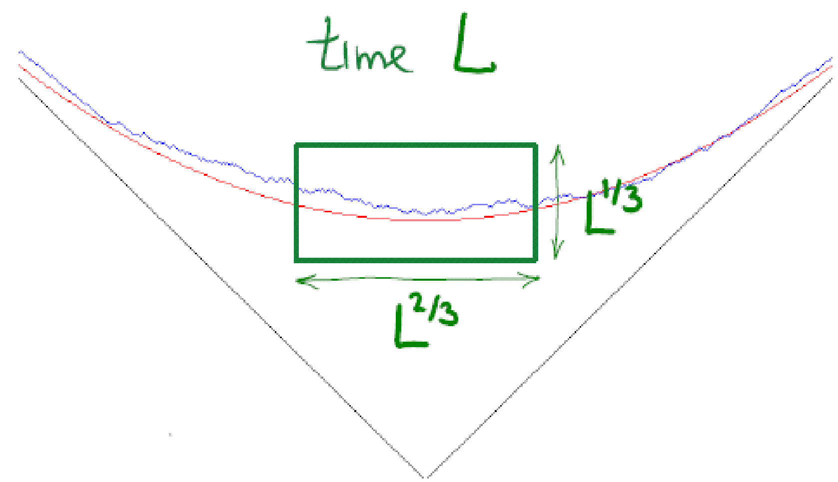
In terms of the average density of particles ρ it is described by the inviscid Burgers equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho(1-\rho)\right)$$

It is known to develop shocks that correspond to traffic jams of the particle system.

TASEP - fluctuations





$$h_L(t,x) := \frac{1}{L^{1/3}}\, h^{TASEP}(Lt, L^{2/3}x) - L^{2/3} \cdot \frac{t}{2}$$

<u>Theorem</u> *[Johansson, 1999]* For TASEP with step initial data

$$\lim_{L\to\infty} \mathbb{P}\{h_L(1,0) \geq -s\} = F_{GUE}(s)$$

Tracy-Widom limit distribution for the largest eigenvalue of large Hermitian matrices

Edge scaling limit of random matrices

Gaussian Unitary Ensemble (GUE) consists of Hermitian $N \times N$ matrices $H = H^*$ distributed as $\text{const} \cdot e^{-\text{Trace}(H^2)} dH$ [Wigner, 1955].

Theorem [Tracy-Widom, 1993]

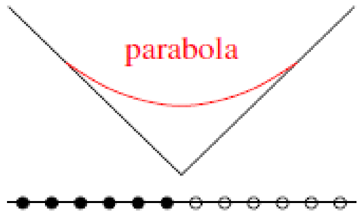
$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{(\text{top eigenvalue of } H) - \sqrt{2N}}{N^{-1/6}} \leq x \right\} = F_{GUE}(x) = F_2(x)$$

where $u = \sqrt{-(\log F_2)''}$ satisfies $u'' = xu + 2u^3$ with appropriate initial conditions.

For real symmetric matrices one similarly defines the Gaussian Orthogonal Ensemble (GOE) and $F_1(x)$.

TASEP – fluctuations depend on hydrodynamic profile

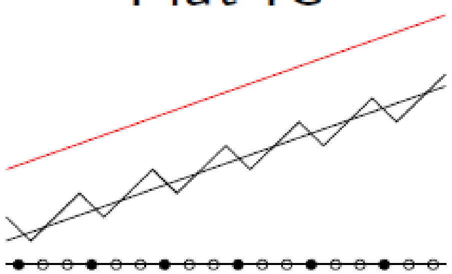
Step IC



$(t^{2/3}, t^{1/3})$-scaling leads to $Airy_2$ process; one-point fluctuations are those of the edge of GUE

Johansson 1999, 2002
Prähofer-Spohn 2002

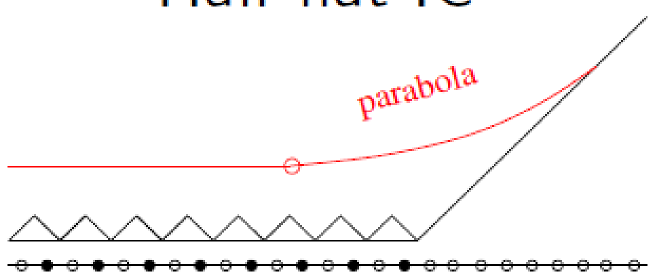
Flat IC



$Airy_1$ process and edge of GOE

Sasamoto 2005
Borodin-Ferrari-Prähofer-Sasamoto 2006, 2007

Half-flat IC



$Airy_1$ in flat part,
$Airy_2$ in curved part,
$Airy_{1\to 2}$ in between.

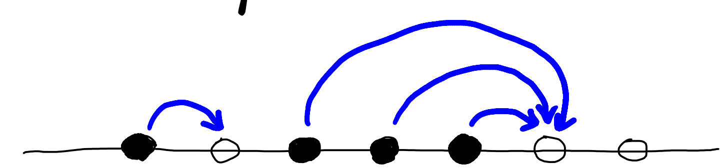
Borodin-Ferrari-Sasamoto 2007

Determinantal integrable particle systems

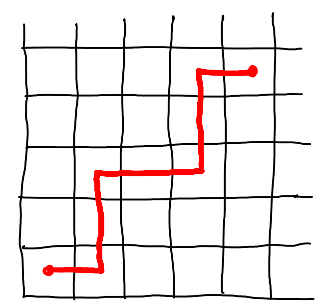
TASEP analyzable via techniques of determinantal point processes (or free fermions, nonintersecting paths, Schur processes).

Other examples include

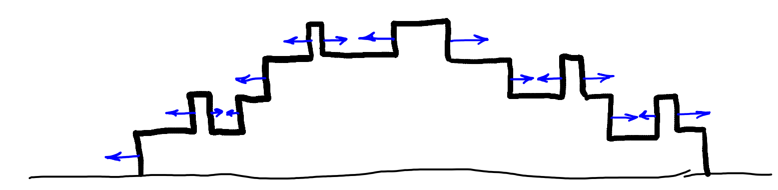
- Discrete time TASEPs with sequential/parallel update
- PushASEP or long range TASEP



- Directed last passage percolation in 2d with geometric/Bernoulli/exponential weights



- Polynuclear growth processes



Breakthrough: Non-determinantal integrable particle systems

- ASEP [Tracy-Widom, 2009], [Borodin-C-Sasamoto, 2012]
- KPZ equation / stochastic heat equation (SHE) [Amir-C-Quastel, 2010], [Sasamoto-Spohn, 2010], [Dotsenko, 2010+], [Calabrese-Le Doussal-Rosso, 2010+], [Borodin-C-Ferrari, 2012]
- q-TASEP [Borodin-C, 2011+], [Borodin-C-Sasamoto, 2012]
- Semi-discrete stochastic heat equation (O'Connell-Yor polymer) [O'Connell, 2010], [Borodin-C, 2011, Borodin-C-Ferrari, 2012]
- Discrete log-Gamma polymer [C-O'Connell-Seppalainen-Zygouras, 2011] [Borodin-C-Remenik, 2012]
- q-PushASEP [Borodin-Petrov, 2013], [C-Petrov, 2013]

GUE corner process

Gaussian Unitary Ensemble (GUE) consists of Hermitian $N \times N$ matrices $H = H^*$ distributed as $\text{const} \cdot e^{-\text{Trace}(H^2)} dH$ [Wigner, 1955].

Call $\lambda_k^{(k)} \le \lambda_{k-1}^{(k)} \le \cdots \le \lambda_1^{(k)}$ eigenvalues of $k \times k$ corner of H.

$$\begin{array}{ccccccccc}
\lambda_N^{(N)} & & \lambda_{N-1}^{(N)} & & \cdots & & \lambda_2^{(N)} & & \lambda_1^{(N)} \\
& \lambda_{N-1}^{(N-1)} & & \lambda_{N-2}^{(N-1)} & & \cdots & & \lambda_1^{(N-1)} & \\
& & \ddots & & \cdots & & . \cdot & & \\
& & & \lambda_2^{(2)} & & \lambda_1^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}$$

$\lambda^{(k)}$ = level k of array

$\lambda^{(k)} \succ \lambda^{(k-1)}$ interlacing levels

GUE corner process

GUE measure on fixed level N [Weyl, Cartan, 1920s]

$$\mathbb{P}(\lambda^{(N)}) = \frac{1}{Z} V(\lambda^{(N)})^2 \prod_{j=1}^{N} e^{-(\lambda_j^{(N)})^2/2}$$

$V(x_1, \ldots x_N) = \prod_{i<j} (x_i - x_j)$

GUE corner process on triangular array [Gelfand-Naimark, 1950]

$$\mathbb{P}(\lambda^{(N)} \succ \lambda^{(N-1)} \succ \cdots \succ \lambda^{(1)}) = \mathbb{P}(\lambda^{(N)}) \cdot \frac{\mathbb{1}\{\lambda^{(N)} \succ \lambda^{(N-1)} \succ \cdots \lambda^{(2)} \succ \lambda^{(1)}\}}{C_N \cdot V(\lambda^{(N)})}$$

Ex: Prove that the volume of the simplex is and compute the constant C_N.

Gibbs property: Given level N, conditional distribution of lower levels is uniform, subject to interlacing condition.

Dyson Brownian motion

Entries of H evolve according to (complex/real) Brownian motions.

The push-forward on $\lambda^{(N)}$ is Markovian DBM [Dyson, 1962]

Generator: $\mathcal{L} = V^{-1} \Delta V$ SDE: $d\lambda_j^{(N)} = B_j + \sum_{i \neq j} \frac{dt}{\lambda_j^{(N)} - \lambda_i^{(N)}}$

Dirichelet Laplacian

But push-forward on entire triangle is NOT Markov!

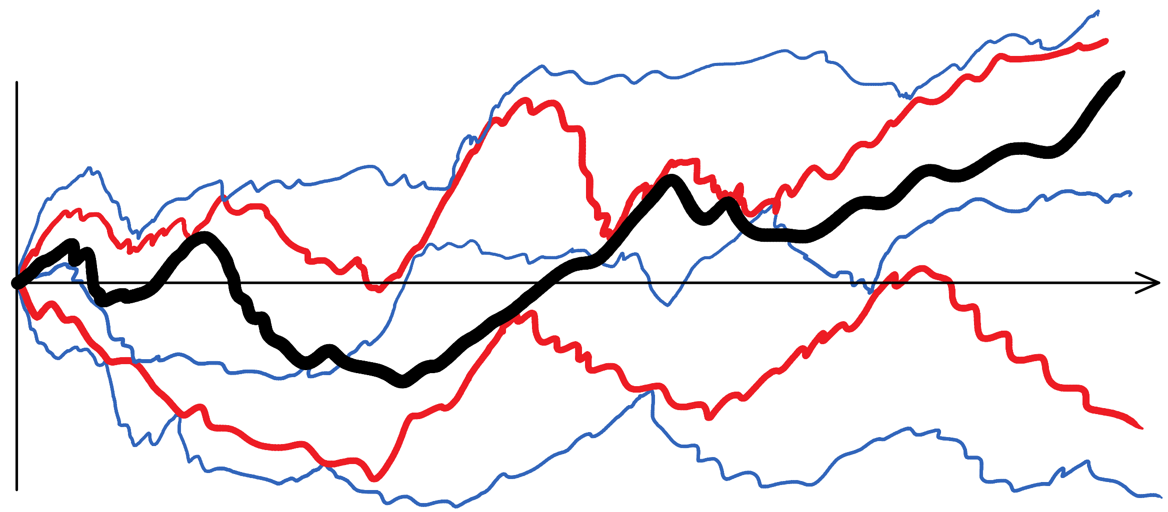
Is it possible to find Markov dynamics on the entire triangle which project on each level to DBM and preserves the Gibbs property?

YES!!! One example is due to [Warren, 2007]

Warren's dynamics: A (2+1)d particle system

Let $\lambda_1^{(1)}$ evolve as a Brownian motion .....

Let $\lambda_k^{(k)}, \ldots, \lambda_1^{(k)}$ evolve as Brownian motions reflected off $\lambda_{k-1}^{(k-1)}, \ldots, \lambda_1^{(k-1)}$

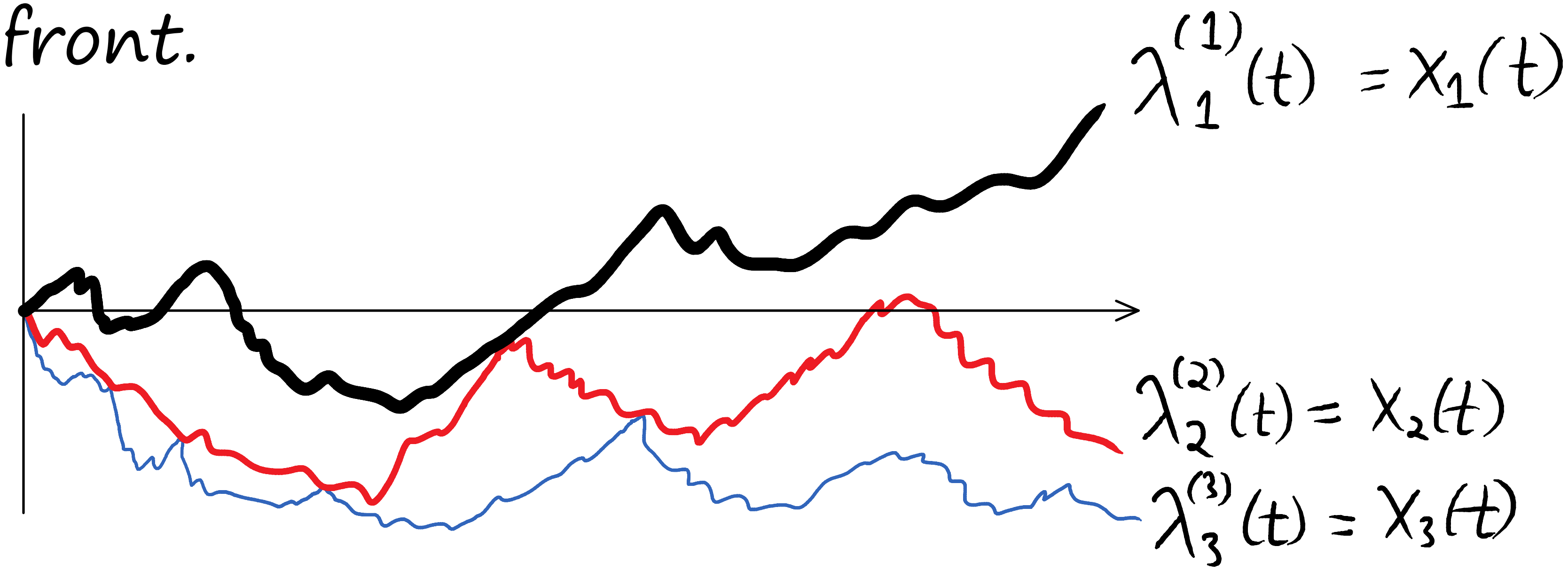


Local dynamic

Preserves Gibbs property on triangle. For instance, start with GUE corner process, run Warren's dynamics and at later time (marginally) end up with rescaled GUE corner process.

Continuous space TASEP

Left-most particles perform a continuous space TASEP in which each particle follows a Brownian motion, reflected off the particle in front.



This is a (1+1)d cut of Warren's (2+1)d dynamics. The GUE corner process is another (2d) cut. Such a coupling explains the occurrence of random matrix type statistics in TASEP

Lecture 1 summary

- Integrable probabilistic system can be characterized by explicit formulas and provides access to large universality classes
- TASEP is such a system in the KPZ universality class
- Dynamics preserving the GUE corners process naturally links continuous space TASEP to GUE

Lecture 2 preview

- Schur measure and process generalize GUE corners process
- Warren's process type dynamics provide link to TASEP
- Determinantal structure leads to explicit formulas /asymptotics